Supplemental Material A

Application of the perturbation and matching technique

Asymptotic solutions

When ϕ^2<< *1,* one notices that Eq. (11) itself suggests the following series as an approximate solution:

$$C^{*} = 1 + A(x^{*})\,\phi^{2} + O(\phi^{4}) \qquad \text{(A-1)}$$

One can also expand *R(C*)* in a Taylor series to obtain:

$$R(C^{*}) = R(1) + R'(1)\,A(x^{*})\,\phi^{2} + O(\phi^{4}) \qquad \text{(A-2)}$$

Here *R'(1)* denotes the first derivative with respect to concentration, which is evaluated at *C* =1*.

By replacing *C** and *R(C*)*, as given by Eqs. (A-1) and (A-2), in Eq. (11) and collecting terms of equal power of ϕ, the following linear ordinary equation is found:

$$\frac{1}{dx^{*}} D_{f}^{*}(x^{*}) \frac{dA}{dx^{*}} = X_{f}^{*}(x^{*}) \qquad \text{(A-3)}$$

which must be solved subject to the following boundary conditions:

$$A = 0 \quad \text{at} \quad x^{*} = 1 \quad \text{and} \quad \frac{dA}{dx^{*}} = 0 \quad \text{at} \quad x^{*} = 0 \qquad \text{(A-4)}$$

In solving Eq. (A-3) with standard analytical methods, and taking into account Eqs. (18) and (19), the following equation is found:

$$A(x^*) = \frac{b\,\Psi^2}{c\,\overline{X}_f\,0.2218}\left[\frac{1}{0.2218}\left[\left(1+\frac{x^*}{\Psi}\right)^{0.2218} - \left(1+\frac{1}{\Psi}\right)^{0.2218}\right] + \ln\left[\frac{\left(1+\frac{1}{\Psi}\right)}{\left(1+\frac{x^*}{\Psi}\right)}\right]\right] \quad \text{(A-5)}$$

By substituting Eq. (A-5) into Eq. (A-2) and then in Eq. (21), the asymptotic expression for the effectiveness factor for the continuum heterogeneous biofilm model (η), is found:

$$\eta = \int_0^1 X_f^*\ R(C^*)\ dx^* = \int_0^1 X_f^*\ \left[1 + R'(1)\ A(x^*)\ \phi^2\right] dx^* \quad \text{(A-6)}$$

In this case:

$$R(C^*) = (\beta+1)\frac{C^*}{(\beta + C^*)} \qquad R(1) = 1 \qquad R'(1) = \frac{\beta}{\beta+1} \quad \text{(A-7)}$$

Then Eq. (24) is found.

When $\phi^2 \to \infty$, the reaction rate is very fast and the rate of diffusion low; therefore, the nutrient is completely consumed at the biofilm-fluid interphase. The dimensionless biofilm density and the nutrient effective diffusivity have values corresponding to $x^* = 1$:

$$X_f^*(x^*) = X_f^*(1) \qquad \text{and} \qquad D_f^*(x^*) = D_f^*(1) \quad \text{(A-8)}$$

Therefore, Eq. (11) yields:

$$D_f^*(1)\ \frac{d^2C^*}{dx^{*2}} = j^{\ 2}\ X_f^*(1)\,R(C^*) \quad \text{(A-9)}$$

Defining:

$$\frac{dC^*}{dx^*} = P \quad \text{(A-10)}$$

$$\frac{d^2C^*}{dx^{*2}} = \frac{dP}{dC^*}\ \frac{dC^*}{dx^*} = P\ \frac{dP}{dC^*} = \frac{1}{2}\ \frac{dP^2}{dC^*} \quad \text{(A-11)}$$

Considering boundary conditions defined in Eq. (16):

$$P(1) = \frac{dC^*}{dx*}\Big|_{x^*=1} \qquad \text{and} \qquad P(0) = \frac{dC^*}{dx*}\Big|_{x^*=0} = 0$$

Then, Eq. (A-9) yields:

$$\frac{dP^2}{dC^*} = \phi^2 \; \frac{2 \; X^*_f(1)}{D^*_f(1)} \; R(C^*) \tag{A-12}$$

Solving Eq. (A-12), the first derivative of C^* at the biofilm-fluid interphase is obtained:

$$P(1)^2 = \left[\frac{dC^*}{dx*}\Big|_{x^*=1}\right]^2 = \phi^2 \; \frac{2 \; X^*_f(1)\,(\beta+1)}{D^*_f(1)} \left[1 + \beta \ln \frac{\beta}{(\beta+1)}\right] \tag{A-13}$$

Thus, with Eqs. (23) and (A-13), the asymptotic expression for the effectiveness factor for large values of ϕ, Eq. (27), is found.

Matching expression for the effectiveness factor

The challenge is to find an expression capable of reproducing Eqs. (24) and (27) when $\phi<<1$ and $\phi>>1$, respectively.

After several attempts (Churchill, 1977; Wedel and Luss, 1980; Gottifredi et al., 1981), Gottifredi and Gonzo (2005) succeeded in finding a rational expression that overcomes the inconvenience presented by previous expressions. The matching equation proposed here is (see Eq. (28)):

$$\eta = \left[\phi^{*2} + \exp(-d \; \phi^{*2})\right]^{-(1/2)} \tag{A-14}$$

After expanding Eq. (A-14) for large and small values of ϕ^*, yields:

$$\eta = \frac{1}{\phi^*} \tag{A-15}$$

And for $\phi^2 \rightarrow 0$

$$\eta \approx 1 - \frac{1}{2}(1-d)\,\phi^{*2} \tag{A-16}$$

By comparing Eqs. (A-15) and (A-16) with Eqs. (24) and (27), respectively, conditions for the unknown parameter (*d*) of Eq. (A-14), are found:

$$d = 1 - 2\,\sigma^* \tag{A-17}$$

with

$$\sigma^* = \sigma\,\rho^2 \tag{A-18}$$